

# Scale-dependent competing interactions: sign reversal of the average persistent current

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The interaction-induced orbital magnetic response of a nanoscale system, modeled by the persistent current in a ring geometry, is evaluated for a system which is a superconductor in the bulk. The interplay of the renormalized Coulomb and Fröhlich interactions is crucial. The diamagnetic response of the large superconductor may become paramagnetic when the finite-size-determined Thouless energy is larger than or on the order of the Debye energy.

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**Introduction.** Renormalization, often accomplished using the renormalization-group method, is one of the basic concepts in physics. It deals with the way various coupling constants (e.g., the electron charge or the Coulomb interaction) change as a function of the relevant scale for the given problem. The scale may be the resolution at which the system is examined, determined by its size and by the relevant energy for the process under consideration. Often, one knows the coupling constant's "bare value" at a less interesting (e.g., a very small or a very large) scale, and what is relevant for experiments is the value at a different, "physical" scale; the coupling on the latter scale is then used for the relevant physics.

An important and particularly interesting application of this idea is in the theory of superconductivity. There, two competing interactions exist: the repulsive Coulomb interaction, starting on the large, microscopic energy scale—typically the Fermi energy/bandwidth  $E_F$ , and the attractive phonon-induced interaction, operative only below the much smaller Debye energy  $\omega_D$ . By integrating over thin shells in momentum (or energy) space [1, 2] one finally gets the variation of any coupling  $g$ , being repulsive or attractive, between a high-energy scale,  $\omega_>$ , and a low one,  $\omega_<$ ,

$$g(\omega_<) = \frac{g(\omega_>)}{1 + g(\omega_>) \ln(\omega_>/\omega_<)} . \quad (1)$$

(We use  $\hbar = c = k_B = 1$ .) Notice that a repulsive/attractive interaction is "renormalized downwards/upwards" with decreasing energy scale. What makes superconductivity possible is that at  $\omega_D$  the repulsion is much smaller than its value on the microscopic scale. Then, at  $\omega_D$  the attraction may win and at lower energies the total interaction increases in absolute value, until it diverges at some small scale, which is just the conventional " $T_c$ " of the given material. The upper panel of Fig. 1 presents this qualitative text-book picture. In this Letter we present quantitative results for a diffusive system whose size  $L$  exceeds the elastic mean-free path  $\ell = v_F \tau_i$  ( $v_F$  is the Fermi velocity and  $\tau_i$  is the mean-free time).

An intriguing situation offered by mesoscopic superconductors, is the possibility to observe the "running", scale-dependent relevant interaction [3]. Obviously the renormalization has to be stopped at the scale corresponding to the system size, which turns out for many physical observables to be the Thouless energy—the inverse of the time it takes an electron to traverse the finite system—which for diffusive systems is  $E_c = D/L^2$  ( $D = v_F \ell/3$  is the diffusion coefficient) [4]. Moreover,  $E_c$  is also the energy scale for many mesoscopic phenomena. Thus, by measuring a quantity which depends on the interaction, as a function of  $L$ , one may deduce the scale-dependence ("running" value) of that interaction. When that quantity is sensitive to the sign of the interaction (e.g., a linear function, up to corrections, of the interaction, often resulting from perturbation theory), one may observe a rather dramatic change of the *sign* of the interaction-dependent term in that quantity!

A unique example for the latter is the orbital magnetic susceptibility of a mesoscopic (or even nanometric) system. Most convenient for calculations is the persistent current [6] flowing in a mesoscopic normal-metal ring in response to an Aharonov-Bohm flux through its opening. The magnetic moment induced by very small fluxes due to these currents is expected to be of the same order of magnitude as that of a small disc made of the same material, and having similar radius and thickness, in response to the same flux.

Moreover, it is well known [4, 5] that the impurity ensemble-averaged persistent current for noninteracting electrons is smaller by typically two orders of magnitude than the interaction-dominated terms. Therefore the latter represents rather well the whole magnetic response of the electrons and its measurement as a function of  $L$  will amount to monitoring the renormalized interaction. This picture is borne out by our calculations. We implement the ideas of Refs. 1 and 2 in the derivation of the superconductive fluctuation-induced partition function, and obtain the effect of the two competing interactions in the Cooper channel. We use this result to study the

average persistent current of a large ensemble of metallic rings in the presence of these interactions.

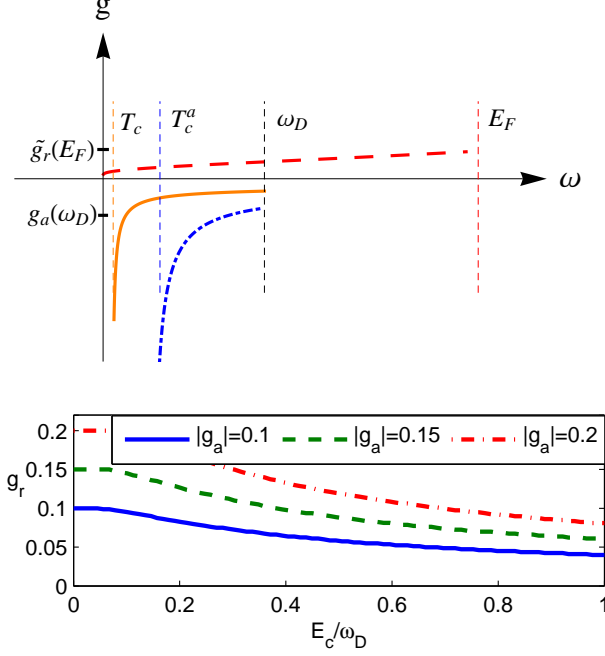


FIG. 1: Upper panel: the renormalized interactions, Eq. (1), as functions of the energy, for a metal which is a superconductor *in the bulk*. When only the attraction exists, its absolute value is renormalized up (with  $\omega_> = \omega_D$ ) until it diverges at  $T_c^a$  (the dash-dotted curve). The repulsion (dashed curve) is renormalized down from its bare value at  $\omega_> = E_F$ . The total interaction starts at  $\omega_D$  from the bare value  $g_a(\omega_D) + g_r(\omega_D)$  and then its absolute value is renormalized up until it diverges at the “true” transition temperature  $T_c$  (full curve). In a mesoscopic system the renormalization stops at the Thouless energy and therefore the relative location of  $E_c$  vs.  $\omega_D$  and the ratio  $|g_a|/g_r(\max\{E_c, \omega_D\})$  will determine the sign of the effective interaction at  $\omega_D$ . The figure is drawn for demonstration purposes with the unrealistic parameter  $E_F/\omega_D = 2.5$ . Lower panel: the lines in the plane  $g_r - E_c/\omega_D$  where the first harmonic of the current changes sign from paramagnetic (above the curve) to diamagnetic (below), for three values of the attraction  $g_a$ . Here and in all figures below,  $T = \omega_D/125 \simeq 2 - 4\text{K}$ , and  $E_F/\omega_D = 25$ .

The renormalization of the interactions has to be carried out down to an energy scale roughly given by  $\max\{T, E_c\}$ , where  $T$  is the temperature. We denote by  $g_a$  the attraction and by  $g_r = [\tilde{g}_r^{-1} + \ln(E_F/\omega_D)]^{-1}$  the repulsion, both at  $\omega_D$ , such that for  $|g_a| > g_r$ , the material in its bulk form is a superconductor. When  $E_c > T$  then depending on the ratios  $E_c/\omega_D$  and  $|g_a|/g_r(\max\{E_c, \omega_D\})$  the resultant interaction may be repulsive or attractive, and the response may be paramagnetic or diamagnetic. These two types of behavior are reflected by the sign of the average current at small fluxes. The lower panel of Fig. 1 presents a “phase di-

agram” showing the regions where the first harmonic of the current is paramagnetic or diamagnetic, as a function of  $E_c/\omega_D$ . Note in particular the existence of a region where although the attraction  $|g_a|$  exceeds the repulsion  $g_r$ , the current is *paramagnetic*, contrary to the usual expectation [5]. The Thouless energy can be increased by making the ring smaller, and/or less disordered.

**Renormalizing the two interactions.** The Hamiltonian we consider consists of a single-particle part,  $H_0(\mathbf{r}) = (-i\nabla - e\mathbf{A})^2/(2m) - \mu + u(\mathbf{r})$ , and a term describing *local* repulsion and attraction, of coupling constants  $\tilde{g}_r \equiv \lambda_r \mathcal{N}/V$  and  $g_a \equiv \lambda_a \mathcal{N}/V$ , respectively,

$$\mathcal{H} = \int d\mathbf{r} \left( \sum_{\sigma} \psi_{\sigma}^{\dagger}(\mathbf{r}) H_0(\mathbf{r}) \psi_{\sigma}(\mathbf{r}) + (\lambda_r - |\lambda_a|) \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) \right), \quad (2)$$

where  $\mathcal{N}$  is the density of states,  $V$  the system volume, and  $\psi_{\sigma}$  ( $\psi_{\sigma}^{\dagger}$ ) destroys (creates) an electron of spin component  $\sigma$  at  $\mathbf{r}$ . Here,  $\tilde{g}_r$  is the bare repulsion,  $\mu$  is the chemical potential,  $u(\mathbf{r})$  is the disorder potential due to nonmagnetic impurities, and  $\mathbf{A}$  is the vector potential, with  $A = \Phi/L$ ,  $\Phi$  giving rise to a persistent current. As the magnetic fields needed to produce such currents are rather small, the Zeeman interaction will be ignored.

The partition function,  $\mathcal{Z}$ , corresponding to the Hamiltonian (2) is calculated by the method of Feynman path integrals, combined with the Grassmann algebra of many-body fermionic coherent states [7]. Introducing the bosonic fields  $\phi(\mathbf{r}, \tau)$  and  $\Delta(\mathbf{r}, \tau)$  via *two* Hubbard-Stratonovich transformations one is able to integrate over the fermionic variables; the result is then expanded to second order in the bosonic fields [8]

$$\mathcal{Z}/\mathcal{Z}_0 = \int D(\Delta, \Delta^*) \int D(\phi, \phi^*) \exp[-\tilde{\mathcal{S}}], \quad (3)$$

with the action  $\tilde{\mathcal{S}}$  given by

$$\tilde{\mathcal{S}} = \sum_{\mathbf{q}, \nu} \left( \beta \mathcal{N} |\Delta_{\mathbf{q}, \nu}|^2 / |g_a| + \beta \mathcal{N} |\phi_{\mathbf{q}, \nu}|^2 / \tilde{g}_r - (\Delta_{\mathbf{q}, \nu} + i\phi_{\mathbf{q}, \nu}) (\Delta_{\mathbf{q}, \nu}^* + i\phi_{\mathbf{q}, \nu}^*) \tilde{\Pi}_{\mathbf{q}, \nu} \right), \quad (4)$$

and  $\mathcal{Z}_0$  denoting the partition function of noninteracting electrons ( $\beta = 1/T$ ). Equation (3) represents the partition function due to superconductive fluctuations, with the action  $\tilde{\mathcal{S}}$ , a sum over the wavevectors  $\mathbf{q}$  and the bosonic Matsubara frequencies  $\nu = 2\pi nT$ , written in terms of the polarization  $\tilde{\Pi}$  averaged over the impurity disorder [9]. It is this object in which the different renormalization of the competing interactions manifests itself.

In a diffusive system, the disorder-averaged polarization is given as a sum over fermionic Matsubara frequen-

cies,  $\omega = \pi T(2n + 1)$ ,

$$\begin{aligned} \tilde{\Pi}_{\mathbf{q},\nu} &= \mathcal{N} \sum_{\omega} \frac{2\pi\Theta[\omega(\omega + \nu)]}{|2\omega + \nu| + Dq^2} = \frac{\mathcal{N}}{T} \sum_{n=0}^{\infty} \frac{1}{n + F_{\mathbf{q},\nu}}, \\ F_{\mathbf{q},\nu} &= 0.5 + (Dq^2 + |\nu|)/(4\pi T), \end{aligned} \quad (5)$$

where  $\Theta$  is the Heaviside function. The first of Eqs. (5) is valid for  $|2\omega + \nu| \ll \tau_i^{-1}$  and  $q\ell \ll 1$ . It represents a divergent sum and therefore a cutoff is needed. The cutoff energy is determined by the relevant interaction. The sums in the terms of Eq. (4) involving  $\Delta$  are bounded by  $\omega_D$  and therefore are cut off at  $n_{\max} = \omega_D/(2\pi T) - 1$ ; these sums are denoted  $\Pi^{\omega_D}$ . The sum multiplying  $|\phi|^2$  is cut off by  $E_F$ , i.e.,  $n_{\max} = E_F/(2\pi T) - 1$ ; this sum is denoted by  $\Pi^{E_F}$ . Therefore, there appear two polarizations, the one cut off by  $\omega_D$  and the other by  $E_F$

$$\frac{T}{\mathcal{N}} \Pi_{\mathbf{q},\nu}^{\omega_D(E_F)} = \Psi\left(\frac{\omega_D(E_F)}{2\pi T} + F_{\mathbf{q},\nu}\right) - \Psi\left(F_{\mathbf{q},\nu}\right), \quad (6)$$

where  $\Psi$  is the digamma function,

Performing the integration over the fields  $\phi$  yields

$$\begin{aligned} \frac{\mathcal{Z}}{\mathcal{Z}_0} &= \prod_{\mathbf{q},\nu} \int D(\Delta_{\mathbf{q},\nu}, \Delta_{\mathbf{q},\nu}^*) \exp\left[-\left(\frac{1}{g_r(\mathbf{q},\nu)} - \frac{1}{|g_a|}\right) \right. \\ &\quad \times \left. \left(\frac{\beta\mathcal{N}}{|g_a| - g_r(\mathbf{q},\nu)} - \Pi_{\mathbf{q},\nu}^{\omega_D}\right) |\Delta_{\mathbf{q},\nu}|^2 \right], \end{aligned} \quad (7)$$

where we have introduced the wavevector- and frequency-dependent renormalized repulsion  $g_r(\mathbf{q},\nu)$

$$g_r^{-1}(\mathbf{q},\nu) = \tilde{g}_r^{-1} + \Psi\left(\frac{E_F}{2\pi T} + F_{\mathbf{q},\nu}\right) - \Psi\left(\frac{\omega_D}{2\pi T} + F_{\mathbf{q},\nu}\right). \quad (8)$$

In the limit  $\omega_D \gg \{Dq^2, T\}$  one may neglect  $F_{\mathbf{q},\nu}$  in the arguments of the digamma functions and retrieve (exploiting the asymptotic form of  $\Psi$ ) the well-known result  $g_r^{-1} = \tilde{g}_r^{-1} + \ln(E_F/\omega_D)$ . In a finite ring, however, the smallest wavevector is of the order  $1/L$  and therefore at low temperatures this procedure is valid only when the Thouless energy is smaller than the Debye energy, i.e., for large enough (and/or clean enough) rings. Carrying out the integrations over the fields  $\Delta$  in Eq. (7) yields

$$\mathcal{Z}/\mathcal{Z}_0 = \prod_{\mathbf{q},\nu} a_{\mathbf{q},\nu}^{-1}, \quad (9)$$

with  $a_{\mathbf{q},\nu} = a(F = F_{\mathbf{q},\nu})$  and

$$\begin{aligned} a(F) &= \frac{1}{|g_a| - g_r} - \Psi\left(F + \frac{\omega_D}{2\pi T}\right) + \Psi(F) \\ &+ \frac{g_r|g_a|}{|g_a| - g_r} \left[ \frac{1}{|g_a|} - \Psi\left(F + \frac{\omega_D}{2\pi T}\right) + \Psi(F) \right] \\ &\times \left[ \Psi\left(F + \frac{E_F}{2\pi T}\right) - \ln\left(\frac{E_F}{\omega_D}\right) - \Psi\left(F + \frac{\omega_D}{2\pi T}\right) \right]. \end{aligned} \quad (10)$$

Equation (9) is our result for the superconductive fluctuation-induced partition function. It is based on the renormalized repulsion, Eq. (8).

**The fluctuation-induced average persistent current.** The thermodynamic persistent current is obtained by differentiating the free energy with respect to the dimensionless flux,  $\varphi = \Phi/\Phi_0$ , i.e.,  $I = (eT)/(2\pi)\partial(\ln \mathcal{Z})/\partial\varphi$  where  $\Phi_0 = h/e$  is the flux quantum. In the ring geometry, the flux enters the longitudinal component,  $q_{\parallel}$ , of the wavevector  $\mathbf{q}$ ,  $q_{\parallel} \equiv q_n = 2\pi(n + 2\varphi)/L$  where  $n$  is an integer [3]. Hence (exploiting the Poisson summation formula)

$$I = \sum_{m=1}^{\infty} \sin(4\pi m\varphi) I_m, \quad (11)$$

where [3]

$$I_m = [(2ieT)/\pi] \sum_{\nu} \int_{-\infty}^{\infty} dx e^{2\pi i x} \partial \ln[a_{x,m,\nu}]/\partial x, \quad (12)$$

with  $a_{x,m,\nu}$  [Eq. (10)] given in terms of  $F_{x,m,\nu} = 0.5 + |\nu|/(4\pi T) + (\pi E_c x^2)/(m^2 T)$  [see Eq. (5)]. The integral in Eq. (12) is handled by a contour integration, and the sign of the result *which determines the magnetic response*, depends on a subtle balance between the contribution of the poles and that of the zeros [3].

It is customary in studies of persistent currents to concentrate on the lowest harmonics, notably on  $I_{m=1}$ , since higher harmonics are exponentially smaller compared to the lowest ones and are also more sensitive to inelastic scattering (dephasing) and pair-breakers (e.g., magnetic impurities) [3]. However, when  $T < E_c$ —values are considered the relative magnitude of harmonics higher than the first one increases, and their contribution should not be discarded.

When only one of the two competing interactions is accounted for, the nature of the magnetic response is dictated by the sign of that interaction, being diamagnetic for attraction and paramagnetic for repulsion [5]. When both interactions are present the picture is changed, as is exhibited in the lower panel of Fig. 1, obtained by numerically calculating  $I_{m=1}$  as a function of the Thouless energy  $E_c$ . The sign of  $I_{m=1}$  depends on the ratio  $E_c/\omega_D$ . As the latter increases, it may change from negative to positive. On the other hand, when  $g_r > |g_a|$  (and the bulk material will not become a superconductor at a finite temperature) the current remains paramagnetic for all values of  $E_c/\omega_D$ .

As mentioned, it is when  $T > E_c$  that the first harmonic represents faithfully the persistent current. Figure 2 depicts the first three harmonics of the current, as functions of the Thouless energy (both normalized by  $\omega_D$ ). The figure indicates that the value of  $E_c$  at which the sign of  $I_m$  is reversed increases with  $m$ . Figure 3 demonstrates the dependence on the strength of the repulsion.

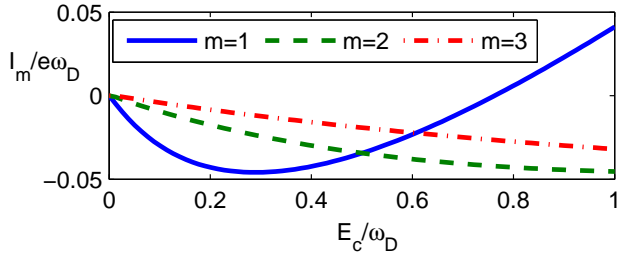


FIG. 2:  $I_m(E_c)$  for  $m = 1, 2$ , and  $3$  (see the legend), with  $|g_a| = 0.2$  and  $g_r = 0.1$ .

As it increases the  $E_c/\omega_D$ -range in which the response is diamagnetic diminishes, and so does also the maximal value of the diamagnetic current. In the paramagnetic-response regime the amplitude of the current increases with  $E_c$  [4]. Figure 4 shows the flux dependence of the current, for two distinct values of  $g_r$ , and displays the possibility for a sign reversal of the magnetic response at low flux. One notes in Fig. 2 that at  $E_c/\omega_D = 0.76$   $I_{m=1}$  is small compared to  $I_{m=2,3}$ ; the dashed curve in Fig. 4 reflects the dominance of the higher harmonics over the first one for relatively small  $g_r$ . For a higher value (solid line there) the first harmonic is dominant.

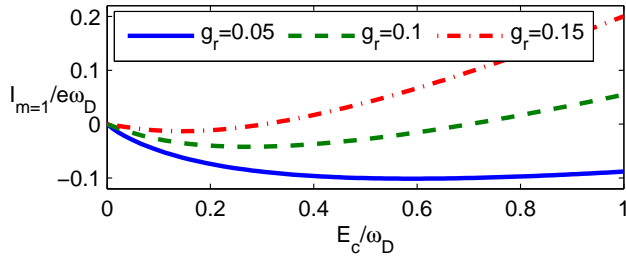


FIG. 3:  $I(E_c)$  for three values of the repulsion and  $|g_a| = 0.2$ .

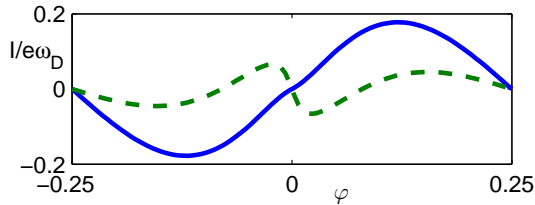


FIG. 4: The current as a function of  $\varphi$ . Solid (dashed) curve is for  $g_r = 0.17$  ( $g_r = 0.1$ ), with  $|g_a| = 0.2$  and  $E_c/\omega_D = 0.76$ .

**Discussion.** We have obtained the wavevector- and frequency-dependent renormalized repulsion, Eq. (8). The two cutoffs on the bare attraction and repulsion,  $\omega_D$  and  $E_F$ , respectively, conspire to determine the sign of the magnetic response. Whereas an overall total at-

traction suffices to lead to a superconducting phase in the bulk material, we find that it does not ensure a diamagnetic response of the mesoscopic system. The reason being that in mesoscopic rings the orbital magnetic response owes its very existence to a finite Thouless energy. Therefore the latter, when large enough can cause the response to be paramagnetic, albeit the strength of the attractive interaction. As the Thouless energy may be controlled experimentally, one may hope that the prediction made in this paper will be put to an experimental test. Relatively large values of persistent currents may be achieved in molecular systems, and small discs are expected to behave similarly at small fluxes. Finally we remark that there may already be experimental indications to the validity of our prediction. Reich *et al.* [10] found that thin enough gold films are paramagnetic. This may well be due to a small grain structure.

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